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Exponentially Fitted Collocation Method For Solving Singular Multi-order Fractional Integro Differential Equations

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Abstract

This work considered the construction of canonical polynomials and used as basis functions for the approximation of singular multi-order fractional integro-differential equations. The idea is that the singular multi-order problem is slightly perturbed with shifted Chebyshev polynomials, and the resulting equation is collocated at equally spaced interior points. The conditions are exponentially fitted with one tau-parameter along with the unknown constants. This results into a system of linear algebraic equations which are then solved using Gaussian elimination method to obtain the unknown parameters involved. Some examples are solved to demonstrate the effectiveness of the method.

Keywords : Canonical Polynomials, Perturbed Collocation Method, Fractional Integro-Differential Equations.

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Introduction

The behaviour of some physical systems in Science and Engineering are governed by singular multi-order fractional integro-differential equations. The equations usually occur as mathematical models, but are later translated into functional equations, which can be of the form

$$D^{\alpha}u(t) + u^{n}(t) + \lambda\sum_{i=0}^{n-1} a_i(t)u^{i}(t) + \int_0^t \sum_{i=0}^{n-1} u^{i}(s)k_i(t,s)ds = f(t) \qquad (1)$$

where, $k_i(t,s) = \frac{1}{(t-s)^p}$ is the kernel, u is the function to be determined, $n \in N$ and $n-1 \leq \alpha \leq n$. The main difficulty is that the singularity behaviour occurs at $t = s$.

Singular multi-order fractional integro-differential equation can be used to describe elasticity mechanism, heat conduction, Dirichlet problems e.t.c., where the need for numerical methods is required in cases the analytical solutions are not known. Hence, the attentions of many researchers have been drawn towards developing appropriate methods that provide the approximate solutions. For instance, Modified Adomian Decomposition Methods was developed by Cheney and Kincaid (2008), Kumar and Singh (2010), Yahaya and Liu (2008), certain classes of Lane emden type equations were solved by Yasir and Zdenek (2012) using the Differential Transform Method. Each authors reported that due to the singularity behaviour of the problems under consideration, the construction of Adomian and Cubic Spline polynomials were difficult to obtain in the Modified Adomian Decomposition and Cubic Spline Methods respectively. In view of this, Canonical polynomials with the aid of Lanczos and Ortiz method are constucted to overcome the drawbacks. Canonical polynomials were also used as basic functions in Taiwo et al. (2014), implementing standard and perturbed collocation methods to solve first and second orders linear integro-differential equations. Other methods which have been considered include the Wavelet method by Wang and Li (2017), in Operational matrix was adopted to reduce the nonlinear Volterra integro-differential equations to a system of algebraic equations; and Collocation method by Avipsita et al. (2017), where Bernstein polynomials were considered as basis functions; where singular boundary value problems and singular initial value problems were solved by the authors respectively. Similarly, fractional order Euler functions were constructed to solve fractional integro-differential equations in Wang et al. (2018). By and large, all the methods were very efficient.

The concept of exponential fitting method has been investigated by many authors, and have individually come up with the idea describing it as a highly efficient method of solution. For instance, Taiwo (2000) investigated the numerical solution of two point boundary value problems by the application of the Cubic Spline Collocation Method using exponential fitting. Among other authors that have also adopted the technique as method of solution are Raptis (1982), Simos (1999), Simos (2006), and Daele and Berghe (2007). Thus, in this work, the exponential fitting with collocation method is introduced to solve equation (1), implementing the constructed canonical polynomials as basis functions.

Definition of Relevant Terms

Definition 1.1: Riemann-Liouville Fractional derivative

Riemann Liouville fractional derivative denoted by D_0^q f(t)is defined as Follows

$$D_0^q f(t) = D_0^m D^{q-m} = D^m J_t^{m-q}; \; q > 0$$

$$= \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_0^x (x-\tau)^{m-q-1} f(\tau) d\tau, q > 0, m-1 < q < m \qquad (2)$$

where, $D^m = \frac{d^m}{dt^m}$ is the derivative part and $D_x^{q-m} f(x) = J_x^{m-q}$is the integral part.

Definition 1.2: Caputo Fractional Derivative

Let $m-1 < q < m$, and $q < 0$, Caputo fractional derivative denoted by

$D_x^q f(x)$ is defined as follows

$$D_x^q f(x) = D_x^{q-m}[D^m f(x)] = J^{m-q}[D^m f(x)]; \qquad m-1 < q < m$$

$$\begin{cases} \frac{1}{\Gamma(m-q)} \int_0^x (x-\tau)^{m-q-1})[D^m f(\tau)] d\tau, & m-1 < q < m \\ \frac{d^m}{dx^m} f(x); q = m, & m \epsilon N \end{cases} \qquad (3)$$

Definition 1.3: Chebyshev Polynomials

The Chebyshev polynomials belong to a family of orthogonal polynomials in the interval [−1,1]. They are widely utilized for their good properties in approximating problems. The Chebyshev polynomial of the first kind of degree ndenoted by Tn(x) and valid in [−1,1] is defined as

$$T_n(x) = \cos\left(n\cos^{n-1}x\right)$$

$$\text{Let } \theta = \cos^{-1}x \tag{4}$$

where $x = \cos\theta$, and the recurrence relation is given as

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 1 \tag{5}$$

But for the purpose of this work, Chebyshev polynomials valid in interval $[a,b]$, is defined as

$$T_n(x) = \cos\left\{n\cos^{n-1}\frac{2x-(a+b)}{b-a}\right\}, n \geq 0$$

$$T_n(x) = \cos\left\{n\cos^{n-1}\frac{2x-(a+b)}{b-a}\right\}T_n(x) - T_{n-1}(x) \tag{6}$$

Methodology

In this section, a general singular multi-order Volterra fractional integro-differential equation of the form:

$$D^{\alpha}y(x) + \sum_{i=0}^{n} p_i y^i(x) + \lambda \int_0 \frac{y(t)}{\sqrt{x-t}}dt = f(x), \quad a \leq x \leq b \tag{7}$$

$$y^k(0) = \alpha_k, k = 0,1,\dots,n-1 \tag{8}$$

Equation (7) is equivalently written as

$$D^{\alpha}y(x) + \sum_{i=0}^{n} p_i y^i(x) = f(x) - \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}}dt, \quad a \leq x \leq b \tag{9}$$

and equation (9) is re-written as

$$D^{\alpha}y(x) + \sum_{i=0}^{n} p_i y^i(x) = g(x) \tag{10}$$

where,

$$g(x) = f(x) - \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}}dt$$

λ and p_i (i≥0) are constant parameters. Also, y(x) and g(x) are functions of x, where x and t are independent variables.

Construction of Canonical Polynomials

Canonical polynomials basis function constructed from equation (7) is obtained as follows:

Expanding equation (10) derived from the general singular and multi-order fractional order integro-differential equations stated in equations (7) and (8), gives

$$\left(D^{\alpha}+p_0+p_1\frac{d}{dx}+p_2\frac{d^2}{dx^2}+\cdots+p_n\frac{d^n}{dx^n}\right)y(x)=g(x) \qquad (11),$$

such that an operator denoted by L is defined as follows

$$L=D^{\alpha}+p_0+p_1\frac{d}{dx}+p_2\frac{d^2}{dx^2}+\cdots+p_n\frac{d^n}{dx^n} \qquad (12)$$

or simply:

$$L\equiv D^{\alpha}+\sum_{i=0}^{n}p_i\frac{d^i}{dx^i} \qquad (13)$$

Following Lanczos (1956), we defined canonical polynomials $Q_j(x), j\geq 0$, which are uniquely associated with the operator L by

$$LQ_j(x)=x^j, j=0,1,2,3,.... \qquad (14)$$

From equation (12), it follows that:

$$Lx^j=\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}x^{j-\alpha}+p_0x^j+p_1jx^{j-1}+p_2j(j-1)x^{j-2}+\cdots+p_nj(j-1)(j-2)\ldots(j-n+1)x^{j-n} \qquad (15)$$

This implies,

$$L[LQ_j(x)]=\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}LQ_{j-\alpha}(x)+p_0LQ_j(x)+p_1jLQ_{j-1}(x)+p_2j(j-1)LQ_{j-2}(x)+\cdots+p_nj(j-1)(j-2)\ldots(j-n+1)LQ_{i-n}(x) \qquad (16)$$

Hence, taking L^{-1} of both sides of equation (16), leads to

$$L^{-1}L[LQ_j(x)]=\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}L^{-1}LQ_{j-\alpha}(x)+p_0L^{-1}LQ_j(x)+p_1jL^{-1}LQ_{j-1}(x)+p_2j(j-1)L^{-1}LQ_{j-2}(x)+\cdots+p_nj(j-1)(j-2)\ldots(j-n+1)L^{-1}LQ_{j-n}(x) \qquad (17)$$

which implies,

$$x^j=\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}Q_{j-\alpha}(x)+p_0Q_j(x)+p_1jQ_{j-1}(x)+p_2j(j-1)Q_{j-2}(x)+\cdots+p_nj(j-1)(j-2)\ldots(j-n+1)Q_{j-n}(x) \qquad (18)$$

Thus,

$$Q_j(x)=\frac{1}{p_0}\left[x^j-\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}Q_{j-\alpha}(x)-p_0Q_j(x)+p_1jQ_{j-1}(x)-p_2j(j-1)Q_{j-2}(x)-\cdots-p_nj(j-1)(j-2)\ldots(j-n+1)Q_{j-n}(x)\right], j\geq 0 \qquad (19)$$

Method of Solution

The perturbed Collocation Method with exponential fitting is considered in this section, as a new method to handle singular and multi-order fractional integro-differential equations.

Here, an assumed approximate solution of the form:

$$y(x) \approx y_N(x) = \sum_{j=0}^{n} a_j Q_j(x), \;\; j \geq 0 \quad (20)$$

is considered, where, $a_j (j \geq 0)$ are constants to be determined, $Q_j (j \geq 0)$ are canonical polynomials constructed in equation (19) and N is the degree of the approximant. Thus, equation (20) is substituted into a slightly perturbed equation (7) to give

$$D^{\alpha} y_N(x) + \sum_{i=0}^{n} p_i {y_N}^{i}(x) + \lambda \int_0^x \frac{y_N(t)}{\sqrt{x-t}} dt = f(x) + H_N(x), \;\; a \leq x \leq b \quad (21)$$

where $H_N(x)$ is the given perturbation term

$$H_N(x) = \sum_{i=1}^{n-1} \tau_i T_{N-i+1}(x) \quad (22)$$

and $T_{N-i+1}(x)$ is the shifted Chebyshev polynomial defined in equation (6). Thus, equation (22) is substituted into equation (21) and further simplification gives

$$D^{\alpha}\left(\sum_{j=0}^{N} a_j Q_j(x)\right) + p_0 \sum_{j=0}^{N} a_j Q_j(x) + p_1 \sum_{j=0}^{N} a_j {Q_j}'(x) + p_2 \sum_{j=0}^{N} a_j {Q_j}''(x) + \cdots$$

$$+ p_n \sum_{j=0}^{N} a_j {Q_j}^{n}(x) + \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = f(x) + \tau_1 T_N(x) + \tau_2 T_{N-1}(x) +$$

$$\tau_3 T_{N-2}(x) + \cdots + \tau_{n-1} T_{N-i+2}(x) \quad (23)$$

where, N is the degree of approximation and n is the highest order singular and multi-order fractional integro differential equation, τ_i,i=1(1)n-1 are the free Tau parameters to be determined.

Hence, equation (23) is further simplified to give

$$\{D^{\alpha}Q_0(x) + P_0Q_0(x) + p_1Q'_0(x) + \cdots + p_nQ_0{}^n(x)\}a_0 +$$

$$\{D^{\alpha}Q_1(x) + P_0Q_1(x) + p_1Q'_1(x) + \cdots + p_nQ_1{}^n(x)\}a_1 +$$

$$\vdots$$

$$\{D^{\alpha}Q_N(x) + P_0Q_N(x) + p_1Q'_N(x) + \cdots + p_nQ_N{}^n(x)\}a_N +$$

$$\lambda\int_0^x \frac{y(t)}{\sqrt{x-t}}dt = f(x) + \tau_1T_N(x) + \tau_2T_{N-1}(x) + \tau_3T_{N-2}(x)$$

$$+ \cdots + \tau_{n-1}T_{N-i+2}(x) \tag{24}$$

The y(t) in equation (24) is evaluated by adopting a Taylor series expansion of appropriate order.

$$\{D^{\alpha}Q_0(x) + P_0Q_0(x) + p_1Q'_0(x) + \cdots + p_nQ_0{}^n(x)\}a_0 +$$

$$\{D^{\alpha}Q_1(x) + P_0Q_1(x) + p_1Q'_1(x) + \cdots + p_nQ_1{}^n(x)\}a_1 +$$

$$\vdots$$

$$\{D^{\alpha}Q_N(x) + P_0Q_N(x) + p_1Q'_N(x) + \cdots + p_nQ_N{}^n(x)\}a_N +$$

$$\lambda\int_0^x \frac{1}{\sqrt{x-t}}\left\{y_N(x) + (t-x)y'_N(x) + \frac{(t-x)^2}{2!}y''_N(x) + \frac{(t-x)^n}{n!}y^n{}_N(x)\right\}dt =$$

$$+\tau_1T_N(x) + \tau_2T_{N-1}(x) + \tau_3T_{N-2}(x) + \cdots + \tau_{n-1}T_{N-i+2}(x) \tag{25}$$

where,

$$y_N(x) = a_0Q_0(x) + a_1Q_1(x) + \cdots + a_NQ_N(x)$$

$$y'_N(x) = a_0Q'_0(x) + a_1Q'_1(x) + \cdots + a_NQ'_N(x)$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$y_N{}^{(n)} = a_0Q_N{}^{(n)}(x) + a_1Q_N{}^{(n)}(x) + \cdots + a_NQ_N{}^{(n)}(x)$$

(26)

Hence, substituting equation (26) into equation (25) and simplifying gives

$$\{D^\alpha Q_0(x) + p_0Q_0(x) + p_1Q'_0(x) + \cdots + p_nQ_0(x) + 2\lambda x^{\frac{1}{2}}Q_0(x) - \frac{2}{3}\lambda x^{\frac{3}{2}}Q'_0(x) + \frac{2}{10}\lambda x^{\frac{5}{2}}Q''_0(x) - \frac{2}{42}\lambda x^{\frac{7}{2}}Q'''_0(x)\}a_0 + \{D^\alpha Q_1(x) + p_0Q_1(x) + p_1Q'_1(x) + \cdots + p_nQ'_1(x) + 2\lambda x^{\frac{1}{2}}Q_1(x) - \frac{2}{3}\lambda x^{\frac{3}{2}}Q'_1(x) + \frac{2}{10}\lambda x^{\frac{5}{2}}Q''_1(x) - \frac{2}{42}\lambda x^{\frac{7}{2}}Q'''_1(x)\}a_1 + \cdots + \{D^\alpha Q_N(x) + p_0Q_N(x) + p_1Q^n{}_N(x) + \cdots + p_nQ^n{}_N(x) + 2\lambda x^{\frac{1}{2}}Q_N(x) - \frac{2}{3}\lambda x^{\frac{3}{2}}Q'_N(x) + \frac{2}{10}\lambda x^{\frac{5}{2}}Q''_N(x) + \frac{2}{42}\lambda x^{\frac{7}{2}}Q'''_N(x)\}a_N = f(x) + \tau_1T_N(x) + \tau_2T_{N-1}(x) + \tau_3T_{N-2}(x) + \cdots + \tau_{n-1}T_{N-i+2}(x) \quad (27)$$

Equation (27) is collocated at point $x = x_i$ to obtain,

$$\{D^\alpha Q_0(x_i) + p_0Q_0(x_i) + p_1Q'_0(x_i) + \cdots + p_nQ_0(x_i) + 2\lambda x_i^{\frac{1}{2}}Q_0(x_i) - \frac{2}{3}\lambda x_i^{\frac{3}{2}}Q'_0(x_i) + \frac{2}{10}\lambda x_i^{\frac{5}{2}}Q''_0(x) - \frac{2}{42}\lambda x_i^{\frac{7}{2}}Q'''_0(x_i)\}a_0 + \{D^\alpha Q_1(x_i) + p_0Q_1(x_i) + p_1Q'_1(x_i) + \cdots + p_nQ'_1(x_i) + 2\lambda x_i^{\frac{1}{2}}Q_1(x_i) - \frac{2}{3}\lambda x_i^{\frac{3}{2}}Q'_1(x_i) + \frac{2}{10}\lambda x_i^{\frac{5}{2}}Q''_1(x_i) - \frac{2}{42}\lambda x_i^{\frac{7}{2}}Q'''_1(x_i)\}a_1 + \cdots + \{D^\alpha Q_N(x_i) + p_0Q_N(x_i) + p_1Q^n{}_N(x_i) + \cdots + p_nQ^n{}_N(x_i) + 2\lambda x_i^{\frac{1}{2}}Q_N(x_i) - \frac{2}{3}\lambda x_i^{\frac{3}{2}}Q'_N(x_i) + \frac{2}{10}\lambda x_i^{\frac{5}{2}}Q''_N(x_i) + \frac{2}{42}\lambda x_i^{\frac{7}{2}}Q'''_N(x_i)\}a_N - f(x_i) + \tau_1T_N(x_i) + \tau_2T_{N-1}(x_i) + \tau_3T_{N-2}(x_i) + \cdots + \tau_{n-1}T_{N-i+2}(x_i)$$

(28)

where,

$$x_i = a + \frac{(b-a)i}{N+2}, \quad i = 1,\ 2,\ \ldots,\ N+1$$

the perturbed collocation method with exponential fitting, an approximation to the conditions is sought by adding $\tau_n e^a$ to equation (8); that is

$$y_N{}^* = \tau_n e^a \qquad (29)$$

and

$$y_N{}^{(k)}(a) + \tau_n e^a = \alpha_k; \qquad k = 0,1,\ldots,n-1 \qquad (30)$$

is obtained. Thus, equations (28) and (30) gives $(N+n+1)$ algebraic equations in $(N+n+1)$ unknown constants $a_0,\ a_1,\ a_2,\ldots,\ a_N, \tau_1, \tau_2, \ldots,\ \tau_n$ to obtain a single polynomial approximation

$$y(x) \approx y_N(x) + y_N{}^*(x); \ \ a \leq x \leq b, \qquad (31)$$

Therefore, equation (31) is the required approximation solution.

Numerical Experiments and Discussion

To illustrate the performance of the presented method, two numerical examples are considered in this section.

Problem 1: Consider a singular multi-order fractional integro-differential equation of the form

Problem 1: Consider a singular multi-order fractional integro-differential equation of the form

$$y'' + \frac{d^{1.5}y(x)}{dx^{1.5}} - \int_0^x \frac{y(t)}{\sqrt{x-t}}dt = 2.2256758334x^{0.5} + 7.221626669x^{2.5} - \frac{16}{15}x^{\frac{5}{2}} - \frac{256}{315}x^{\frac{9}{2}} + 2 + 12x^2, \quad 0 \leq x \leq 1 \qquad (32)$$

subject to the conditions

$$y(0) = 0, y'(0) = 0 \qquad (33)$$

The exact solution is given as $y(x) = x^4 + x^2$

Problem 2: Consider a singular multi-order fractional integro-differential equation of the form

$$y''' + \frac{d^{2.75}y(x)}{dx^{2.75}} + xy(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}}dt = 6 + 12x^2 + \frac{32}{35}x^{\frac{7}{2}} + 6.619575908x^{0.25}, \quad 0 \le x \le 1 \qquad (34)$$

subject to the conditions

$$y(0) = 0,\ y'(0) = 0,\ y''(0) = 0 \qquad (35)$$

The exact solution is $y(x) = x^3$

Tables of Results and Graphical Representations

Table (i) :

Absolute Errors of Problem 1 for case $N = 8$

x	Exact Solution	Approximate Solution	Absolute Error
0.000	0.00000000	0.000000000	0.0000000
0.100	0.01010000	0.010099998	1 .1291E −09
0.200	0.04160000	0.041599998	2.0010E −09
0.300	0.09810000	0.098099973	2.6614E −08
0.400	0.18560000	0.185599959	4.0101E −08
0.500	0.31250000	0.312499957	4.2119E −08
0.600	0.48960000	0.489599950	4.9991E −08
0.700	0.73010000	0.730099950	5.0010E −08
0.800	1.04960000	1.049599581	4.1935E −07
0.900	1.46610000	1.466099500	4.9989E −07
1.000	2.00000000	1.999999437	5.6292E −07

Table (ii):
Absolute Errors of Problem 2 for case $N = 10$

x	Exact Solution	Approximate Solution	Absolute Error
0.000	0.0000000	0.000000000	0.0000000
0.100	0.0010000	0.000999999	5.8350*E* - 11
0.200	0.0080000	0.007999999	5.9828*E* - 10
0.300	0.0270000	0.026999999	6.1135*E* - 10
0.400	0.0640000	0.063999999	6.3596*E* - 10
0.500	0.1250000	0.124999992	7.5421*E* - 09
0.600	0.2160000	0.215999992	7.9915*E* - 09
0.700	0.3430000	0.342999991	8.3044*E* - 09
0.800	0.5120000	0.511999997	2.3153*E* - 09
0.900	0.7290000	0.728999998	1.8460*E* - 09
1.000	1.0000000	0.999999962	3.7380*E* - 08

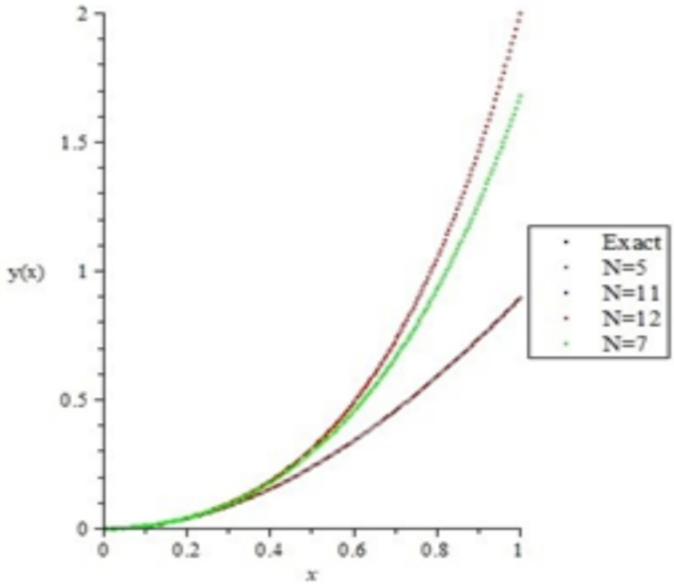

Figure 1: The Graph of the Approximate and Exact Solution for Problem 1

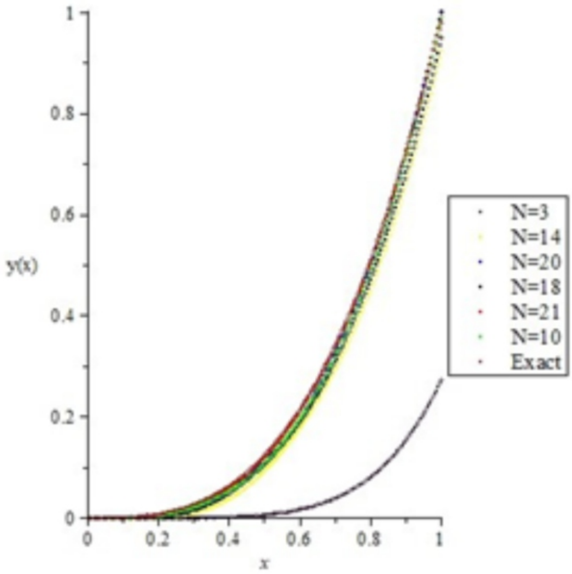

Figure 2: The Graph of the Approximate and Exact Solution for Problem 2

Discussion of Results

It is required to obtain approximate solutions of singular and multiorder fractional integro-differential equations due to difficulties in obtaining the analytical solutions. Thus, exponentially fitted collocation method by canonical polynomials is considered as the method of solution in this work, and it is revealed that it's highly efficient and effective in terms of convergence as it's clearly seen that the approximate solution converges rapidly to the exact solution as the degree of the approximant N increases.

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